Adaptive steady-state stabilization for nonlinear dynamical systems

David J. Braun*
Department of Mechanical Engineering, Vanderbilt University, Nashville, Tennessee 37235, USA
(Received 22 February 2008; revised manuscript received 13 June 2008; published 29 July 2008)

By means of LaSalle's invariance principle, we propose an adaptive controller with the aim of stabilizing an unstable steady state for a wide class of nonlinear dynamical systems. The control technique does not require analytical knowledge of the system dynamics and operates without any explicit knowledge of the desired steady-state position. The control input is achieved using only system states with no computer analysis of the dynamics. The proposed strategy is tested on Lorentz, van der Pol, and pendulum equations.

DOI: 10.1103/PhysRevE.78.016213 PACS number(s): 05.45.Gg

I. INTRODUCTION

Controlling nonlinear dynamical phenomena is one of the most important parts of modern engineering. Traditionally, the control strategy has been strongly related to the analytical model of system dynamics; however, for many natural processes a satisfactory analytical model is hard to define. This is the prototypical case in biology, physiology, and medicine as well as in natural systems where chaotic response characterizes the experimental data [1]. Motivated by this challenge, the intention here is to construct a *model-independent* controller for both chaotic and nonchaotic dynamical systems [2,3].

The idea of controlling chaotic dynamics was presented by Ott, Grebogy, and York (OGY) [4]. The OGY controller uses the *butterfly effect* of chaotic systems to control the long-time dynamics with carefully chosen small perturbations. This method does not require any *a priori* analytical knowledge of the system dynamics, which makes it attractive for experimental implementation [5,6].

The failure of the OGY strategy under the impact of nonlinearity has well-documented experimental verification [7]. The reason for this failure comes from the linearization argument used in the OGY algorithm, which is not valid in the neighborhood of a nonhyperbolic desired state [8]. Any linearization can be understood as a source of instability because it restricts the stable attraction region subjected to the controlled state. In this sense, to provide *practical stability* (stability with a wide enough attraction region), it is often necessary to base the control design on strong nonlinear arguments [9].

Controlling chaotic (and nonchaotic) dynamics has also been considered by other methods. In this light, a general adaptive control scheme was introduced by Huberman and Lumer [10] and further developed by Sinha *et al.* [11,12]. Chaos control, by means of variable thresholding, was both analytically and experimentally explored in [13–17]. The famous delayed-feedback controller proposed by Pyragas [18,19] has been shown as particularly well suited to stabilize unstable fixed points and unstable periodic orbits for large class of dynamical systems.

Traditionally, the goal of chaotic control has been to stabilize unstable periodic orbits embedded in a chaotic attractor. In this paper, however, it is preferred to stabilize unstable fixed points. This goal can always be accepted if the periodic or chaotic behavior means erosion of the system performance.

In many real situations, the position of the desired steady state is not known *a priori*. In all these cases, the control strategy has to locate this state adaptively. Such a problem of adaptive stabilization around the unknown steady state (or periodic orbit) was recently considered in [20–22].

The intention herein is to construct a model-independent controller to stabilize unstable steady states for nonlinear dynamical systems. The control algorithm is based on a closed-loop controller with variable feedback gain which operates with the help of an adaptive steady-state estimator. The proposed controller does not require any computer analysis of the system dynamics and works without knowing the system steady-state position. Using LaSalle's invariance principle [23], the controller is tailored to stabilize unstable fixed points with minimalistic assumptions on the underlying system dynamics.

The idea presented can be seen as an extension of [24,20–22]; namely, it neither requires knowledge of the steady-state position nor depends on the control gain selection.

II. CONTROLLER

Consider the nonlinear dynamical system modeled by

$$\dot{x} = f(x), \tag{1}$$

where $t \in [0,\infty)$ is an independent variable, $x \in R^n$ are phase coordinates, and $\dot{x}=dx/dt$ are phase velocities, while $f:D \to R^n$ is a time-invariant vector field defined on $D \subseteq R^n$. Assume that the system has at last one fixed point $x^* \in D$, while f is smooth enough (i.e., Lipschitz on D) to guarantee the existence and uniqueness of the solution $x(t)$ with any initial condition $x_0 \in D$.

Suppose that the physical system considered is complicated, possibly chaotic, and we are not able to reconstruct a reasonable analytical model (1) to describe its behavior. Moreover, let us assume that although an equilibrium state x^* exists, its position is not known for the control design. Under these circumstances, our goal is to stabilize a dynamical evolution of (1) around x^*.

*david.braun@vanderbilt.edu

Adding control perturbations u to the right-hand side of (1), the controlled system reads

$$\dot{x}_i = f_i(x) + u_i, \quad i \in \{1,2,\ldots,n\}. \tag{2}$$

The simplest way to obtain the adaptive control u is to make it proportional to the distance between the unstable fixed point and the system's present state $u \sim (x - x^*)$. Unfortunately, implementation of this perturbation cannot be realized without explicit knowledge of the steady-state position x^*. To overcome this limitation, instead of x^*, its estimated value should be used to build an alternative control law. The estimated position of the fixed point can be obtained from the first-order estimator

$$\dot{y}_i = \lambda_i(x_i - y_i), \tag{3}$$

where $y \in R^n$ and $\lambda \in R^n_+$.

Let us assume that x^* is an asymptotically stable equilibrium state of the controlled system (2), which means that any solution $x(t)$ of (1) starting close enough to x^* remains near x^* and moreover $x \to x^*$, in long-time evolution. This property guarantees that for any $\lambda_i > 0$ the solution of (3) is bounded and moreover $y \to x^*$ [25]. We would like to point out that in context of this paper, Eq. (3) should be understood as a steady-state estimator [22], rather than a state estimator usually used in control theory. In this light, it is not a primary requirement from y to track x as good as possible, but rather to approach x^* in long-time evolution. By means of the beginning assumption of this paragraph, in order to use y instead of x^*, it is necessary to show that x^* is an asymptotically stable equilibrium of the controlled system (2). In the following, the control strategy will be tailored with the aim to achieve the mentioned stability requirement for x^*.

There are a large number of control algorithms used in nonlinear dynamics. The proportional feedback, derivative control, and time delay feedback [18] are all well-studied methods with their own advantages and limitations [26]. Here, motivated by [24], the following closed-loop controller with variable feedback gain $k_i = k_i(t)$ is proposed:

$$u_i = -k_i(x_i - y_i), \quad \dot{k}_i = \alpha_i(x_i - y_i)^2, \tag{4}$$

where $\alpha_i \in R_+$. With zero initial gain $k(0) = 0$, each component in $k(t) \in R^n_+$ is a nondecreasing positive function which tends monotonically to its maximum value when the desired state is reached, $x \to x^* \Rightarrow k \to k^* = \{k_{1\,\max}, k_{2\,\max}, \ldots, k_{n\,\max}\}$. On the other hand, because $u \sim (x - y)$, the controller does not change the position of the original fixed point, which means that it operates without a steady-state error.

Consider now the controlled dynamics with the aim of showing that $x \to x^*$ as $t \to \infty$. Let V be a real-valued function:

$$V = \frac{1}{2}\sum_{i=1}^{n}(x_i - y_i)^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\alpha_i}(L - k_i)^2, \tag{5}$$

where L is a positive constant parameter. The time derivative of V along the controlled trajectory (2)–(4) reads

$$\dot{V} = \sum_{i=1}^{n}(x_i - y_i)f_i(x) - (L + \lambda_i)(x_i - y_i)^2. \tag{6}$$

Since $f(x)$ is locally Lipschitz, it is bounded on its domain D, which implies $\exists l < \infty$ such that $\forall i, |f_i(x)| \leq l|x_i - y_i|$ for $\forall x \in D$ if $x_i \neq y_i$. Now, for all states for which $\exists i, x_i \neq y_i$, we can choose $L > l$ in (6) to obtain the inequality

$$\dot{V} \leq W = -(L - l)\sum_{i=1}^{n}(x_i - y_i)^2 \leq 0. \tag{7}$$

Notice that (7) is valid, not just for $x_i \neq y_i$, but rather for $\forall x \in D$; namely, if $\forall i, x_i = y_i$, then $\dot{V} = 0$, which simply follows from (6).

By means of LaSalle's invariance principle, (7) secures that for $t \to \infty$, any bounded solution of (1) tends to the largest invariant set of (2)–(4) contained in the set $E = \{(x, y, k) : x = y\}$ for which $\dot{V} \leq 0$. However, the largest invariant set $M = \{(x, y, k) : x = y = x^*, k = k^*\}$ of (2)–(4) only contains the equilibrium state x^* of the original equation (1), and therefore, the (unstable) fixed point of the plant becomes asymptotically stable under the control influence.

Finally, the nonlinear controlled process reads

$$\dot{x}_i = f_i(x) - k_i(x_i - y_i),$$
$$\dot{y}_i = \lambda_i(x_i - y_i),$$
$$\dot{k}_i = \alpha_i(x_i - y_i)^2. \tag{8}$$

In order to secure asymptotic behavior of the controlled solution toward x^*, the uncontrolled solution of (1) needs to be bounded. Practically, if there exists a bounded closed set Ω such that $x^* \in \Omega$ and every solution of (1) starting from $x_0 \in \Omega$ remains for all future time in this set, then the controlled equilibrium point x^* becomes asymptotically stable, while Ω is the region of its asymptotic stability. One of the easiest consequences of this property is that every stable system (1) becomes asymptotically stable under the presented controller (2)–(4). On the other hand, if the uncontrolled system has a globally stable periodic orbit or a global chaotic attractor, then [all solutions of (1) are bounded and] the originally unstable equilibrium point x^* becomes globally asymptotically stable under (8).

By nature of the estimator dynamics, (3) always chooses a fixed point of an original system for the reference target. Generally if (1) in addition to a fixed point has another invariant set [for example, a periodic orbit or chaotic attractor] the algorithm eliminates it from consideration, and hence that set is noninvariant for the controlled system (2)–(4). On the other hand, if (1) has two or more fixed points $M = \{x^*_I, x^*_{II}, \ldots\}$, then (8) elects one to stabilize it. In this sense, the control strategy is persistent under the presence of multiequilibrium, although we may not know which particular state will be stabilized.

According to the global construction, (8) provides stabilization for any $\alpha, \lambda > 0$. However, these parameters have a strong influence on the stabilization time and transient behavior. Particularly, if $\lambda \gg 1$, the estimator output behaves as

$y \approx x$, producing slow control adaptation, and similar, slow control dynamics is expected for $\alpha \ll 1$. On the other hand, for $\lambda \ll 1$, the estimator spends a long time locating the fixed point, due to its slow dynamics, while for $\alpha \gg 1$ the fast dynamics of the control gain might not be realizable with the limitations of real actuators. All these effects produce time-consuming stabilization, which should be overcome in practical implementations.

The nonlinear argument, used in the presented controller design, makes it possible to obtain stabilization even for near-nonhyperbolic steady states [8]. However, nonhyperbolicity often produces different time scales in the underlying dynamics, dividing the state coordinates into fast and slow ones. Let x_s be one observed slow variable, and using (8) let us construct a one-dimensional controller: $\dot{k}_s = -\alpha(x_s - y_s)^2$ with $\dot{y}_s = -\lambda(y_s - x_s)$. Naturally, this slow variable, $\dot{x}_s \approx 0$, mimics a quasi-steady-state producing long-time transients due to the slow control and estimator adaptation, $\dot{y}_s \approx 0$, $\dot{k}_s \approx 0$. In this light, the control strategy based on the slow-state coordinates produces an unpractical response which is the reason why this type of system variable should be eliminated to obtain good performance for the practical control scheme.

When state coordinate measurement is provided, electronic implementation of the controller dynamics in (8) would require a simple *RC* circuit by means of a low-pass filter for the steady-state estimation y and an integrator to obtain k. The squared error function $(x_i - y_i)^2$, which is necessary to obtain the control gain, can be implemented using an integrated-circuit multiplier. The multiplier, which does not add a particular cost to the implementation, is not required for a constant-gain adaptive scheme [22] and was also not needed for threshold-based chaos control [16].

Finally, let us discuss the optimality of the algorithm presented. Generally, the methods to construct an optimal controller [27,28] depend on analytical knowledge of the system dynamics $f(x)$. In practice, however, optimality always erodes due to the neglected imperfections in analytical modeling. This is because the optimality is a system-dependent concept and the mathematically optimal strategy is suboptimal in the best practical cases. Without any knowledge about the model dynamics, the controller presented could be optimized through its tunable parameters α and λ. On the other hand, inspired by the OGY method, the ergodic behavior of the chaotic systems can be used as an energy-saving mechanism. With ergodic behavior, the system trajectory, at a certain time, becomes near to the desired state without any control input. When it happens, the controller starts to operate, keeping the trajectory in the desired state with only small perturbations. In this sense, the chaotic response provides an opportunity for energy-saving control [29].

III. APPLICATION TO PHYSICAL SYSTEMS

In this section the controller presented is tested on physical problems. In this light, steady-state control for Lorentz and van der Pol equations is presented. In addition, application to the periodically excited, pendulum equation shows that the controller can be used for some nonautonomous dynamical systems. In all examples presented, the dimension of the proposed controller is reduced.

A. Lorentz chaos

Consider the famous Lorentz equations [30]

$$\dot{x}_1 = \sigma(x_2 - x_1),$$

$$\dot{x}_2 = rx_1 - x_2 - x_1 x_3,$$

$$\dot{x}_3 = x_1 x_2 - bx_3, \tag{9}$$

where σ, b, and r are positive parameters related to turbulent convection for which (9) was originally derived. For $r > 1$, this system has three equilibrium points: the trivial saddle point $x_0^* = (0,0,0)$ and two additional symmetric equilibria, $x_\pm^* = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. For $r > \sigma(\sigma+\beta+3)/(\sigma-\beta-1)$, all equilibrium points become unstable while the system exhibits chaotic behavior. To stabilize these unstable points, instead of full state control, only one input $u_2 = -k(x_2 - y)$ is added to the second equation in (9), where the feedback gain and the estimated position of the equilibrium point y are obtained from (8).

The complete set of equations for the controlled Lorentz system reads

$$\dot{x}_1 = \sigma(x_2 - x_1),$$

$$\dot{x}_2 = rx_1 - x_2 - x_1 x_3 - k(x_2 - y),$$

$$\dot{x}_3 = x_1 x_2 - bx_3,$$

$$\dot{k} = \alpha(x_2 - y)^2,$$

$$\dot{y} = \lambda(x_2 - y). \tag{10}$$

Setting, $\sigma = 10$, $b = 8/3$, $r = 28$, $\alpha = 0.2$, $\lambda = 0.2$, and $(x_1, x_2, x_3, k, y)_{t=0} = (1,1,2,0,0)$ and turning the controller on at $t = 50$, Eqs. (10) are stabilized at $x_-^* = (-6\sqrt{2}, -6\sqrt{2}, 27)$, Fig. 1.

If instead of $t = 50$ the controller is turned on at $t = 52$, the solutions starting from the same initial position tend to the another equilibrium state $x_+^* = (6\sqrt{2}, 6\sqrt{2}, 27)$. Along this property, the proposed strategy possesses some flexibility to change the operational point by simply turning the controller on and off. Generally, the only thing one needs to do is to wait until the uncontrolled state x becomes close to the expected equilibrium x^* and then turn the controller on. In most cases this simple strategy secures stabilization around x^*.

Our numerical experiments showed that, if one particular input shows a dominant effect under the rest input sate, then the controller dimension may significantly be reduced. More explicitly, the stabilization algorithm can in some cases be based only on one (dominant) control input reducing the dimension of the original scheme from $(x, y, k) \in R^{3n}$ to $(x, y, k) \in R^{n+2}$. This dimensional reduction may have a high practical importance, providing simpler controller construction.

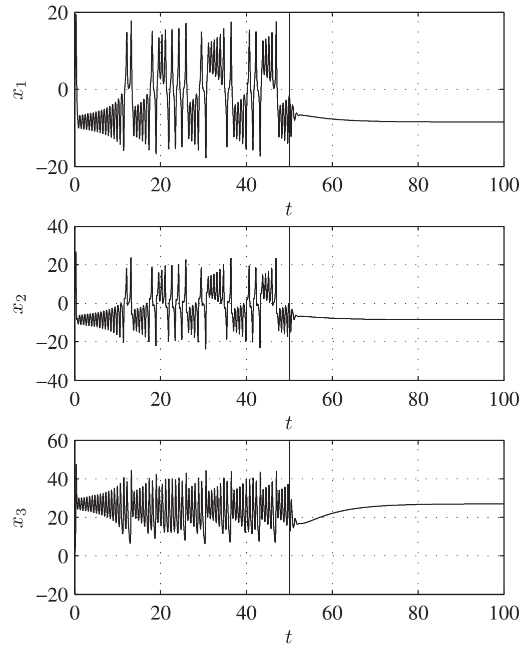

FIG. 1. Stabilization of the unstable focus x_{-}^{*} for the Lorentz system. Uncontrolled chaotic evolution $t \in [0,50)$, controlled dynamics $t \in [50,100)$.

Unlike $x_{\pm}^{*}$, which are unstable focuses, we were not able to stabilize the trivial saddle point x_0^* with stable steady-state estimator $\lambda > 0$. However, as was recognized by Pyragas [20], saddle points could be stabilized with an unstable estimator, $\lambda < 0$. Motivated by this idea, we have applied an unstable steady-state estimator, $\lambda = -0.2$ in (10). Figure 2 shows the saddle point stabilization.

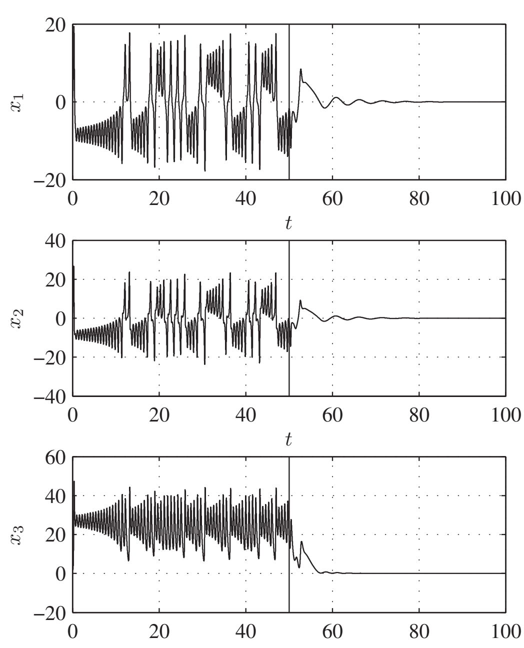

FIG. 2. Stabilization of the unstable saddle point x_0^* for the Lorentz system. Uncontrolled chaotic evolution $t \in [0,50)$, controlled dynamics $t \in [50,100)$, $\alpha = 0.2$, and $\lambda = -0.2$.

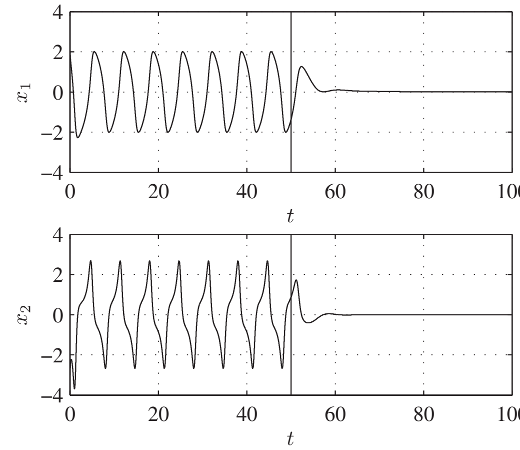

FIG. 3. Steady-state stabilization of the van der Pol oscillator; uncontrolled evolution on the limit cycle $t \in [0,50)$, controlled dynamics $t \in [50,100)$.

Note that $\lambda < 0$ introduced an unbounded degree of freedom in (8) which is not supported by the stability proof presented. In this light, saddle stabilization, although possible with $\lambda < 0$, is not proved as generally valid in the present context. In electronic implementations, the unstable estimator would require an *RC* circuit with negative resistance [20].

B. van der Pol oscillator

The intention herein is to inspect the behavior of the presented control algorithm in vicinity of a stable periodic attractor. In this light, consider now the van der Pol nonlinear oscillator

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0. \tag{11}$$

This equation has one unstable trivial state $(x_1, x_2) = (0,0)$ and one globally asymptotically stable limit cycle which attract all trajectories from the phase space. In order to stabilize (11) around the equilibrium position, the following control scheme is proposed:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 - k(x_2 - y),$$

$$\dot{y} = \lambda(x_2 - y),$$

$$\dot{k} = \alpha(x_2 - y)^2. \tag{12}$$

Setting $(x_1, x_2, k, y)_{t=0} = (2, -3, 0, 0)$, $\epsilon = 1$, and $\alpha = 0.5$, $\lambda = 0.1$, the stabilization is clearly shown in Fig. 3.

C. Controlling pendulum

Although the proposed controller is built for autonomous systems $\dot{x} = f(x)$, it can be used for some time-varying cases $\dot{x} = f(t, x)$. The theoretical background of this nonautonomous extension lies in the fact that LaSalle's invariance principle is applicable not just for autonomous, but for all nonaoutonomous systems with time-invariant positive limit set (periodic, almost periodic, asymptotically autonomous, etc., systems) [31]. The implementation is shown in the following example.

Consider the dimensionless equation of the mathematical pendulum,

$$\ddot{x} + \beta\dot{x} + (1 + p\cos\omega t)\sin x = 0, \tag{13}$$

where $x(t)$ is the angular displacement measuring from the vertical axis and β characterize viscous dissipation, while the pendulum pivot is subject to the vertically periodic forcing $p\cos\omega t$. Depending on the excitational parameters (p,ω), this nonlinear oscillator can undergo various types of intriguing behavior. Setting $\beta=0.1$, $\omega=2$, and $p=2$, the solution of (13) becomes locally unstable, but globally bounded, characterizing the chaotic regime. Under these circumstances, we are going to stabilize the vertical position of (13) defined by the multiple equilibrium states $x^*=\{0,\pm\pi,\pm2\pi,\ldots\}$.

Using (8), the control scheme becomes

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -\beta x_2 - (1 + p\cos\omega t)\sin x_1 - k(x_2 - y),$$

$$\dot{y} = \lambda(x_2 - y),$$

$$\dot{k} = \alpha(x_2 - y)^2, \tag{14}$$

where instead of the full state control, only one input is proposed. Choosing $\alpha=0.25$, $\lambda=0.5$, and $(x_1,x_2,y,k)_{t=0}=(1,1,0,0)$ the pendulum is stabilized around $x^*=(2\pi,0)$, Fig. 4.

Generally, if a multiequilibrium points exist, the position of the stabilized state is strongly dependent on the system dynamics, initial conditions, and controller parameters. Although the controller guarantees stabilization for systems with multiequilibrium, it is not known *a priori* which particular equilibrium will be stabilized. This type of a so-called, multiequilibrium uncertainty can be overcome, setting $y(0)=x^*$ and $\lambda=0$. It is important to mention that if x^* is predefined, then there is no longer a need for the state estimator (3), while the controller reduces to that presented by Huang [8].

In numerical experiments with (14), the control strategy kept the privilege to stabilize $\{0,\pm2\pi,\ldots\}$, overcoming the saddle states $\{\pm\pi,\pm3\pi,\ldots\}$. In this case, application of the idea presented by [22] may be an alternative method to stabilize the saddle equilibrium. In the present context, however, the analytical form of the system dynamics is assumed to be unknown, and as such, the feedback gain selection in [22] would be only possible through a hand-tuning process.

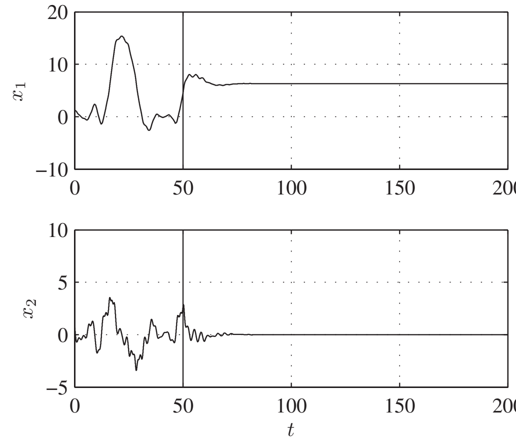

FIG. 4. Steady-state stabilization of the periodically excited pendulum; uncontrolled evolution $t\in[0,50)$, controlled dynamics $t\in[50,200)$.

Finally, it is important to mention that the idea presented is developed under general assumptions and as such is valid for a wide class of dynamical systems. Further relaxation by means of a general nonlinear form of the controlled dynamics, $\dot{x}=f(t,x,u)$, and possible extension to incorporate stabilization of unstable periodic orbits, although nontrivial by means of the stability proof, may further extend the applicability of the proposed idea.

IV. CONCLUSION

In summary, a control strategy is proposed with the aim of stabilizing an unstable steady state embedded in a nonlinear dynamical system. The operation of the adaptive controller with variable feedback gain (4) is combined with the first-order steady-state estimator (3). The resulting stabilization strategy does not require knowledge about the system's steady state, and moreover it works without using any analytical form of the system dynamics. Compared to traditional controllers, the idea presented might have advantages as long as there is no clear understanding of the dynamic system which is under control.

ACKNOWLEDGMENTS

The author would like to express his gratitude to Dr. Michael Goldfarb, Dr. Božidar D. Vujanović, and Dr. Alvin M. Strauss for their help and encouragement.

[1] L. Glass, Nature (London) **410**, 277 (2001).

[2] D. J. Christini and J. J. Collins, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **44**, 1027 (1997).

[3] V. Petrov, A. Haaning, K. A. Muehlner, S. J. Van Hook, and H. L. Swinney, Phys. Rev. E **58**, 427 (1998).

[4] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).

[5] W. L. Ditto, S. N. Rauseo, and M. L. Spano, Phys. Rev. Lett. **65**, 3211 (1990).

[6] A. Garfinkel, M. Spano, W. Dito, and J. Weiss, Science **257**, 1230 (1992).

[7] W. van de Water and J. de Weger, Phys. Rev. E **62**, 6398

(2000).
[8] D. Huang, Phys. Rev. Lett. **93**, 214101 (2004).
[9] H. K. Khalil, *Nonlinear Systems* (Prentice-Hall, Englewood Cliffs, NJ, 2002).
[10] B. A. Huberman and E. Lumer, IEEE Trans. Circuits Syst. **37**, 547 (1990).
[11] S. Sinha, R. Ramaswamy, and J. S. Rao, Physica D **43**, 118 (1990).
[12] S. Sinha and N. Gupte, Phys. Rev. E **58**, R5221 (1998).
[13] S. Sinha and D. Biswas, Phys. Rev. Lett. **71**, 2010 (1993).
[14] S. Sinha, Phys. Rev. E **49**, 4832 (1994).
[15] L. Glass and W. Zeng, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 1061 (1994).
[16] K. Murali and S. Sinha, Phys. Rev. E **68**, 016210 (2003).
[17] C. Wagner and R. Stoop, J. Stat. Phys. **106**, 97 (2002).
[18] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[19] K. Pyragas, Phys. Lett. A **206**, 323 (1995).
[20] K. Pyragas, Phys. Rev. Lett. **86**, 2265 (2001).
[21] K. Pyragas, V. Pyragas, I. Z. Kiss, and J. L. Hudson, Phys. Rev. Lett. **89**, 244103 (2002).
[22] K. Pyragas, V. Pyragas, I. Z. Kiss, and J. L. Hudson, Phys. Rev. E **70**, 026215 (2004).
[23] J. P. LaSalle, Proc. Natl. Acad. Sci. U.S.A. **46**, 363 (1960).
[24] D. Huang, Phys. Rev. E **73**, 066204 (2006).
[25] L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations* (Springer-Verlag, Berlin, 1959).
[26] A. L. Fradkov and R. J. Evans, Annu. Rev. Control **29**, 3356 (2005).
[27] R. Bellman, *Dynamic Programming* (Princeton University Press, Princeton, NJ, 1957).
[28] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes* (John Wiley and Sons, New York, 1962).
[29] T. Shinbrot, C. Grebogi, E. Ott, and J. A. Yorke, Nature (London) **363**, 411 (1993).
[30] E. N. Lorentz, J. Atmos. Sci. **20**, 130 (1963).
[31] J. P. LaSalle, Nonlinear Anal. Theory, Methods Appl. **1**, 83 (1976).